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THE MOTION OF A POINT MASS ALONG A STRING†

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As a supplement to results obtained earlier [1], the general integral of motion of a point mass along a string is determined, and the influence of friction is evaluated. © 2001 Elsevier Science Ltd. All rights reserved.

The problem of the motion of a point mass along a string was considered in [1] as an example of a mechanical system for which Zhukovskii's method [2] can be used to determine the particular integral of motion.

1. THE GENERAL INTEGRAL OF MOTION

We will consider, as previously [1], the plane motion of a point mass (a bead) along an elastic weightless thread (a string) stretched between two fixed points. In the plane of motion we will fix a stationary system of coordinates Oxy so that the points where the string is fastened lie symmetrically on the x axis at a distance l from the origin of coordinates.

Let r_1 and r_2 be the distances from the bead to the right-hand and left-hand fastening points, respectively, let g be the coefficient of tensile stiffness of the string, and let Δ be the preliminary tension of the string. Then, the potential energy of the string, V, is given by the formula

$$V = \frac{1}{2} g[(r_1 + r_2 + \Delta - 2l)^2 - \Delta^2] \tag{1.1}$$

Here the misprint in [1] has been corrected.)

This problem is similar to the well-known problem of two fixed centres [3].
However, to separate the variables, the introduction of new time is also required.

After the introduction of the dimensionless coordinates

$$q_1 = \frac{r_1 + r_2}{2l}, \quad q_2 = \frac{r_1 - r_2}{2l} \quad (-1 \leqslant q_2 \leqslant 1 \leqslant q_1 \leqslant \infty)$$

the potential energy, the kinetic energy, and the Hamilton–Jacobi equation will respectively take the form

$$V = \frac{1}{2} g[(2lq_1 + \Delta - 2l)^2 - \Delta^2]$$

$$T = \frac{1}{2} l^2 m(q_1^2 - q_2^2)\left(\frac{\dot{q}_1^2}{q_1^2 - 1} + \frac{\dot{q}_2^2}{1 - q_2^2}\right)$$

$$(q_1^2 - 1)\left(\frac{\partial W}{\partial q_1}\right)^2 + (1 - q_2^2)\left(\frac{\partial W}{\partial q_2}\right)^2 = 2l^2 m(h - V)(q_1^2 - q_2^2)$$

where h is the energy constant.

To separate the variables, we will introduce a new time according to the formula

$$dt = \sqrt{q_1^2 - q_2^2}\, d\tau \tag{1.2}$$

This replacement is one-to-one only when $q_1 = 1$ and $|q_2| = 1$ (at the points where the string is fastened).

The Hamilton–Jacobi equation becomes

$$(q_1^2 - 1)\left(\frac{\partial W}{\partial q_1}\right)^2 + (1 - q_2^2)\left(\frac{\partial W}{\partial q_2}\right)^2 = -l_1 q_1^2 + l_2 q_1 + l_3$$

$$l_1 = 4gml^4, \quad l_2 = 4mgl^3(2l - \Delta), \quad l_3 = 2ml^2(h - 2gl(l - \Delta))$$

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Assuming $W = W_1(q_1) + W_2(q_2)$, we obtain the system of equations

$$(q_1^2-1)\left(\frac{\partial W_1}{\partial q_1}\right)^2 = -l_1 q_1^2 + l_2 q_1 + l_3 - c_1,\quad (1-q_2^2)\left(\frac{\partial W_2}{\partial q_2}\right)^2 = c_1$$

where c_1 is an arbitrary constant ($c_1 > 0$). We will write its solution

$$W = \int\sqrt{\frac{l_3 + l_2 q_1 - l_1 q_1^2}{q_1^2 - 1}}\,dq_1 + \sqrt{c_1}\arcsin q_2$$

and the solution for the generalized coordinates and momenta

$$\frac{\partial W}{\partial c_1} = C_1,\quad \frac{\partial W}{\partial h} = \tau - C_2,\quad \frac{\partial W}{\partial q_1} = p_1,\quad \frac{\partial W}{\partial q_2} = p_2$$

where C_1 and C_2 are arbitrary constants, or in expanded form

$$q_2 = \sin\left[\sqrt{c_1}\left(\frac{\tau + C_2}{l^2 m} - C_1\right)\right] \tag{1.3}$$

$$\frac{dq_2}{d\tau} = \frac{\sqrt{c_1}\sqrt{1-q_2^2}}{l^2 m},\quad t = \int_0^\tau \sqrt{q_1^2(\tau) - q_2^2(\tau)}\,d\tau$$

$$I \equiv \int[(q_1^2-1)(l_3 + l_2 q_1 - l_1 q_1^2)]^{-1/2}dq_1 = \frac{\tau + C_2}{l^2 m}$$

$$\frac{dq_1}{d\tau} = \sqrt{\frac{(q_1^2-1)(l_3 + l_2 q_1 - l_1 q_1^2)}{l^2 m}}$$

When investigating the motions of a point close to the x axis, the quantity $u^2 = q_1 - 1$ is close to zero (the product $q\Delta$ is large). Therefore, with some error, it is possible to simplify the elliptic integral

$$I = 2\int\left[(2+u^2)\left(l_4 - \frac{l_1}{l}\Delta u^2 - l_1 u^4\right)\right]^{-1/2} du \approx$$

$$\approx \int(2l_4 - l_5^2 u^2)^{-1/2} du = \frac{1}{l_5}\arcsin\frac{l_5 u}{\sqrt{2 l_4}}$$

$$l_4 = 2l^2 mh - c_1,\quad l_5^2 = 8l^3 mg\Delta - 2l^2 mh + c_1$$

and to write an approximate solution for q_1 in explicit form

$$q_1 = \frac{2l_4}{l_5^2}\sin^2\frac{l_5(\tau + C_2)}{2l^2 m} + 1$$

2. THE STRENGTH OF THE STRING

An important characteristic of the system, from the viewpoint of strength, is the maximum tension of the string $r_1 + r_2 - 2l$, an estimate of which can be obtained from an estimate of the range of possible motion

$$g[(r_1 + r_2 - 2l + \Delta)^2 - \Delta^2] < 2h$$

For maximum tension, it is also possible to provide an estimate in the case of a sudden deceleration of a point on the string

$$V_0 = \frac{2El^2(x - a_0)^2 + y^2}{(l_2 - a_0^2)(2l - \Delta)} + E[2l - (r_1 + r_2)] < h$$

where E is the modulus of elasticity of the string, related to its unit cross-section area and a_0 is the abscissa of a heavy point fixed on the string in the rest position. An expression for the potential energy V_0 can be obtained, for example, by moving a heavy point fixed on the string from the rest point to the point $(x, 0)$ along the x axis, and then moving it parallel to the y axis to the point (x, y) at which the deceleration occurred.

3. THE INFLUENCE OF DRY FRICTION ON THE MOTION OF A POINT

For applications, it may also be of interest to take into account dry friction during the motion of a point along the string.

The tension forces of the string

$$Q = \frac{\partial V}{\partial q_1} = g(2lq_1 - 2l + \Delta)$$

acting on a heavy point along the directions r_1 and r_2 create an equivalent force directed along the bisector of the angle α between the directions r_1 and r_2:

$$F = 2Q\cos\frac{\alpha}{2} = 4gl(2lq_1 - 2l + \Delta)\sqrt{\frac{q_1^2 - 1}{q_1^2 - q_2^2}}$$

which generates a friction force μF, where μ is the coefficient of dry friction. The latter largely prevents any change in the coordinate q_2 (during motions close to the x axis, i.e. when $q\Delta$ is large).

We will write the Lagrange equation for this system

$$\frac{q_1^2 - q_2^2}{q_1^2 - 1}\ddot{q}_1 + \frac{q_2^2 - 1}{(q_1^2 - 1)^2}q_1\dot{q}_1^2 - \frac{2q_2}{q_1^2 - 1}\dot{q}_1\dot{q}_2 + aq_1 - b = 0 \tag{3.1}$$

$$\frac{q_1^2 - q_2^2}{1 - q_1^2}\ddot{q}_2 + \frac{q_1^2 - 1}{(1 - q_2^2)^2}q_2\dot{q}_2^2 - \frac{2q_1}{1 - q_2^2}\dot{q}_1\dot{q}_2 = -2\mu(aq_1 - b)\sqrt{\frac{q_1^2 - 1}{q_1^2 - q_2^2}}\operatorname{sign}\dot{q}_2$$

$$a = \frac{4g}{m},\quad b = \frac{2g(2l - \Delta)}{lm}\quad (a > b > 0)$$

Regarding μ as a small parameter on sections where $\dot{q}_2$ is of constant sign, and taking into account the presence of a complete integral of the generating system, according to perturbation theory it is possible to find a solution for the perturbed system by differentiation operations and by taking quadratures. Here, very lengths expressions arise, and it is more convenient to use the method of successive approximations.

Making a replacement of variables, which is used on changing to Hamilton's equations, we transform system (3.1) to the form

$$\dot{q}_1 = \frac{(q_1^2 - 1)p_1}{q_1^2 - q_2^2},\quad \dot{q}_2 = \frac{(1 - q_2^2)p_2}{q_1^2 - q_2^2},\quad \dot{p}_1 = -\frac{(1 - q_2^2)q_1p_1^2}{(q_1^2 - q_2^2)^2} - aq_1 + b \tag{3.2}$$

$$\dot{p}_2 = \frac{(q_1^2 - 1)q_2p_2^2}{q_1^2 - q_2^2} - 2\mu(aq_1 - b)\sqrt{\frac{q_1^2 - 1}{q_1^2 - q_2^2}}\operatorname{sign}\dot{q}_2$$

where p_1 and p_2 are generalized momenta.

Suppose that, close to the right-hand support, a velocity impulse v was imparted to the heavy point in a direction similar to the direction on the left-hand support, i.e.

$$q_1 = 1 + \varepsilon_1,\quad q_2 = -1 + \varepsilon_2,\quad \dot{q}_1 = \varepsilon_3,\quad \dot{q}_2 = v \quad \text{when } t = 0$$

where ε_1, ε_2 and ε_3 are small quantities and $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. It will also be necessary to determine the time t^* before the point stops ($\dot{q}_2(t^*) = 0$) and the coordinate $q_2(t^*)$ if it turns out that $t^* < t_1$, where t_1 is the time of motion of the point up to the specified neighbourhood of the left-hand support when there is no friction. In this case, the factor sign $\dot{q}_2$ in system (3.2) will be replaced by unity.

Taking as the zero approximation the generating solution q_{10}, q_{20}, p_{10} and p_{20}, for a first approximation we obtain

$$q_{11} = q_{10},\ q_{21} = q_{20},\ p_{11} = p_{10},\ p_{21} = p_{20} - p_3$$

$$p_3 = 2\mu\int_0^t (aq_{10} - b)\sqrt{\frac{q_{10}^2 - 1}{q_{10}^2 - q_{20}^2}}dt$$

Using the fact that the explicit form of q_{10} and q_{20} as a function of τ is known, and taking into account relation (1.2), we obtain

$$p_3 = 2\mu\int_0^t (aq_{10} - b)\sqrt{q_{10}^2 - 1}d\tau \tag{3.3}$$

In the second approximation

$$q_{22} = \int_0^t \frac{1 - q_{20}^2}{q_{10}^2 - q_{20}^2} p_{21} dt$$

From relations (3.2) it can be seen that $\dot{q}_{22} < p_{20} - p_3$. If the equation $p_3 = p_{20}$ in relation to t has a positive solution, smaller than t_1, it will approximately represent the quantity t^*.

The expression for p_3 after integration is fairly lengths. Therefore, for p_3 we will make an upper estimate. Since $q_1 \leqslant q_{1a}$, where $q_{1a} = 2l_4/l_5^2 + 1$ is the amplitude value of the solution q_{10}, it follows that

$$p_3 < 2\mu(aq_{1a} - b)\sqrt{q_{1a}^2 - 1}t \quad и \quad t^* \cong \upsilon\left[2\mu(aq_{1a} - b)\sqrt{q_{1a}^2 - 1}\right]^{-1}.$$

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